# THE GROWTH OF A LAMINAR BOUNDARY LAYER ON A FLAT PLATE SET IMPULSIVELY INTO MOTION

## (RAZVITIE LAMINARNOGO POGRANICHNOGO SLOIA NA PLASTINKE, PRNEDENNOI IMPUL'SIVNO V DVIZHENIE)

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1. We will consider the anomalous boundary-layer motion which develops when a semi-infinite flat plate initially at rest in a viscous incompressible fluid begins abruptly to move parallel to itself with a constant velocity $U_0$. The problem of finding such a flow may be formulated in the usual way [1], with the motion transformed by considering the plate as motionless and the fluid as moving with the velocity $U_0$ at infinity, in terms of an integration of the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.1}$$

under the conditions that

$$\begin{aligned} u &= U_0 \quad &&\text{for } y > 0,\ x = 0,\ t \geqslant 0 \ \text{ or } \ x \geqslant 0,\ t = 0 \\ u = v &= 0 &&\text{for } y = 0,\ x > 0,\ t \geqslant 0 \\ u &= U_0 &&\text{for } y \to \infty,\ x \geqslant 0,\ t \geqslant 0 \end{aligned} \tag{1.2}$$

Here $xoy$ is an orthogonal system of coordinate axes whose origin is at the leading edge of the plate and whose $x$-axis is directed along the plate parallel to the velocity of the oncoming stream; $t$ is the time; $\nu$ is the coefficient of kinematic viscosity; and $u$ and $v$ are the projections of the velocity at any point in the boundary layer on the axes of $x$ and $y$ respectively.

In analysing possible methods for solving the problem as posed here it is necessary first of all to point out one special feature, which is that the method of successive approximations [2] commonly used in the solution of problems of nonstationary boundary layers does not lead to a correct result in the present case. This fact follows at once from an inspection of the structure of the first approximation, obtained as a result of discarding the convective terms in (1.1). This approximation evidently corresponds to the development of a flow near a flat plate infinite in both directions and thus in independent of $x$. In addition, succeeding approximations are all found to be identical with the first. Since the motion should reflect the influence of the leading edge of the plate on the formation of the boundary layer and consequently should depend on $x$, no useful solution is obtained.

Application of other approximate methods of boundary-layer calculation [3,4] *et al.**) has revealed certain novel features of the difficulty in the present problem. For example, an approximate solution based on the method of Shvets [4] calls for the integration of the equation

$$\frac{\partial \varphi}{\partial x} + \frac{8}{3}\frac{1}{U_0}\frac{\partial \varphi}{\partial t} = 16\,\nu U_0 \tag{1.3}$$

where $\delta(x,t)$ is the thickness of the boundary layer and $\phi = U_0\delta^2$. In addition, the function $\phi$ is required on physical grounds to satisfy the conditions

$$\varphi = 0 \quad \text{for } t = 0, \qquad \varphi = 0 \quad \text{for } x = 0$$

The solution which is obtained is

$$\delta = \begin{cases} \sqrt{6\nu t}, & x > \frac{3}{8} U_0 t \\ \sqrt{16\nu x / U_0}, & 0 \leqslant x < \frac{3}{8} U_0 t \end{cases} \tag{1.4}$$

Hence it is seen that two regions, separated from each other by a moving rectilinear boundary at $x = 3/8\ U_0 t$, exist near the plate. On one side of this boundary ($x > 3/8\ U_0 t$) there is a non-stationary motion which is unaffected by the leading edge of the plate, and on the other ($0 < x < 3/8\ U_0 t$) there is a stationary motion corresponding to the solution of the Blasius problem. As time goes on the stationary regime spreads toward larger values of $x$ and gradually occupies the whole plate.

The solution represented by (1.4) has the defect that the first partial derivatives of the function $\delta(x,t)$ are discontinuous at $x = 3/8\ U_0 t$. This in turn implies a discontinuity in the first partial derivatives of the velocity component $u$ at $x = 3/8\ U_0 t$, and also a discontinuity in the velocity component $v$. This result is explained by the fact that (1.4) was constructed as an integral surface of (1.3) passing through the two mutually perpendicular straight lines $x = 0$ and $t = 0$ in the space $(x, t, \phi)$. The necessity for just such a solution is dictated by the initial value problem (1.1), (1.2) and is evidently connected with the peculiarity already noted.

Similarly, the construction of an "outer" solution, valid in some

* In particular, an aerodynamic method for the computation of non-stationary boundary layers was developed by Struminskii [5], and the problem of the plate was treated by way of example.

measure near the edge of the boundary layer, leads to an expression of the form

$$u = \begin{cases} U_0 \operatorname{erf} \dfrac{y}{2\sqrt{\nu t}}, & x > U_0 t \\ U_0 \operatorname{erf} \dfrac{y}{2}\sqrt{\dfrac{U_0}{\nu x}}, & 0 \leqslant x < U_0 t \end{cases} \qquad \left(\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\alpha^2} d\alpha\right) \qquad (1.5)$$

which follows if the multiplicative factor $u$ is replaced by $U_0$ in the transport terms of (1.1), and $\partial u/\partial y$ is put equal to zero. The same conclusions can be drawn from (1.5) as from the formulas (1.4). The only difference between (1.4) and (1.5) is found in the velocity of propagation of the boundary between the stationary and nonstationary motions. This velocity is equal to 3/8 $U_0$ in formula (1.4) and to $U_0$ in (1.5).

It must be concluded that the results enumerated above are a consequence of the incorporation of the boundary-layer theory in the simplifie mathematical formulation of the original problem in the form (1.1), (1.2) In reality, during the first instant of time after the plate has begun to move in the fluid the influence of the leading edge can only make itself felt for small values of $x$, for which the local Reynolds numbers $U_0 x/\nu$ are not large. This influence is subsequently propagated downstream towar larger values of $x$. Thus the formation of the flow near $x = 0$ in the firs moment of time plays an important role in the later development of the boundary layer on the plate. To obtain a correct picture of the flow in this region it is necessary to turn to a solution of the full Navier-Stokes equations, inasmuch as the statement of the problem for small $U_0 x/\nu$ in the form (1.1), (1.2) exhibits the same deficiencies as in the case of stationary motion [6]. In the light of the remarks already made, we will now proceed to investigate the initial period of development of the flow on a semi-infinite plate through an integration of the Navier-Stokes equations.

2. The Navier-Stokes equations may be reduced to a single equation for the stream function $\psi$ in the form

$$L(\psi) = K(\psi) \qquad (2.1)$$

where

$$L(\psi) = \nu\Delta\Delta\psi - \frac{\partial}{\partial t}\Delta\psi$$
$$K(\psi) = \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\Delta\psi - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\Delta\psi \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

We will look for a solution of (2.1) with the aid of the method of successive approximations. It is known that in the first moment of time the motion of the fluid near the plate will be potential. For very small $t$, therefore, the stream function $\psi$ may be taken as equal to $\psi_0 = U_0 y$. On substituting this function $\psi_0$ in the right-hand side of the equation

(2.1), we will arrive at an equation $L(\psi_1) = 0$. Differential equations for succeeding approximations may be obtained in a similar way. The method indicated for solving (2.1) is equivalent to the representation of the stream function $\psi$ by a series

$$\psi = \psi_0 + \psi_1 + \ldots + \psi_{n-1} + \psi_n + \ldots \tag{2.2}$$

the terms of which satisfy the following system of equations:

$$\begin{aligned}
&\psi_0 = U_0 y\\
&L(\psi_1) = K(\psi_0)\\
&L(\psi_2) = K(\psi_0 + \psi_1) - K(\psi_0)\\
&\ldots\ldots\ldots\ldots\ldots\\
&L(\psi_n) = K(\psi_0 + \psi_1 + \ldots + \psi_{n-1}) - K(\psi_0 + \psi_1 + \ldots + \psi_{n-2})\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{2.3}$$

We will limit the investigation to the first approximation. Noting that $u_1 = \partial\psi_1/\partial y$, we can write down the equation for the first approximation to the velocity component $u_1$:

$$\triangle\left(\frac{\partial u_1}{\partial t} - \nu\triangle u_1\right) = 0 \qquad \text{или} \quad \frac{\partial u_1}{\partial t} = \nu\triangle u_1 \tag{2.4}$$

At the same time, physical considerations which are not confined to the framework of the boundary-layer theory require the boundary and initial conditions to be prescribed in the following manner:

$$\begin{aligned}
&u_1 = 0 && \text{for } t = 0\\
&u_1 = -U_0 && \text{for } y = 0,\ x > 0\\
&u_1 \to 0 && \text{for } |y| \to \infty,\ x > 0\\
&u_1 \to 0 && \text{for } |x| \text{ or } |y| \to \infty,\ x < 0
\end{aligned} \tag{2.5}$$

Further, putting

$$\psi_1 = U_0\sqrt{\nu t}\,\varphi_1(\eta, \xi), \qquad \eta = \frac{y}{\sqrt{\nu t}}, \qquad \xi = \frac{x}{\sqrt{\nu t}} \tag{2.6}$$

the problem (2.4), (2.5) when written in terms of the two independent variables $\eta, \xi$ becomes

$$\begin{aligned}
&-\frac{1}{2}\left(\eta\frac{\partial u_1}{\partial\eta} + \xi\frac{\partial u_1}{\partial\xi}\right) = \triangle u_1 && \left(\triangle = \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2}\right)\\
&u_1 = 0 && \text{for } \eta \to \infty,\ \xi \to \infty\\
&u_1 = -U_0 && \text{for } \eta = 0,\ \xi > 0\\
&u_1 \to 0 && \text{for } |\eta| \to \infty,\ \xi > 0\\
&u_1 \to 0 && \text{for } |\eta| \text{ ıorı } |\xi| \to \infty,\ \xi < 0
\end{aligned} \tag{2.7}$$

Taking into consideration (2.3) and the expression (2.6) for $\psi_1$, the series (2.2) may be written in the form

$$\psi = U_0 y + U_0 \sqrt{\nu t}\,\varphi_1(\eta, \xi) + U_0^2 t \varphi_2(\eta, \xi) + U_0 t\,[U_0\, \varphi_{31}(\eta, \xi) + \\ + U^2 \sqrt{\frac{t}{\nu}}\, \varphi_{32}(\eta, \xi) + \frac{U_0^3}{\nu}\, \varphi_{33}(\eta, \xi)] + \dots \qquad (2.8)$$

Thus it is seen from (2.8) that construction of a solution of the equation (2.1) by the method of successive approximations is nothing else but a search for a solution of (2.1) in the form of a series in powers of the time $t$.

In investigating the development of the flow near the plate in the initial period of the motion we will restrict ourselves to the first two terms in the expansion (2.8) with the understanding that $U_0\sqrt{\nu t} \gg U_0^2 t$, or $U_0^2 t/\nu \ll 1$. Our problem will then consist of finding solutions of (2.4), (2.5). It should be noted that the equation (2.4) and the conditions (2.5) are quite often encountered in problems of a similar nature [7,8]. In particular, Howarth [8] has investigated the development of the fluid motion near a semi-infinite plate ($y = 0$, $x > 0$) which at the instant $t = 0$ begins to move with a velocity $U_0$ parallel to its lateral edge. By tracing the analogy between the mathematical formulation of the problem of Howarth [8] and the problem (2.4), (2.5), the velocity component $u$ may be accurately represented up to the second approximation in the following two ways.

In the form of a series written in polar coordinates

$$\frac{u}{U_0} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{2n-1}{4}\right)}{(2n-1)\,\Gamma\left(\frac{2n-1}{4}\right)} \left(\frac{R}{2}\right)^{n-1/2} e^{-1/4 R^2} F_1\left(\frac{2n+3}{4}, \frac{2n+1}{4}, \frac{R^2}{4}\right) \sin\left(n - \frac{1}{2}\right)\theta \qquad (2.9)$$

where $R = \sqrt{\eta^2 + \xi^2}$ and $\theta = \operatorname{arc\,tg}\,(\eta/\xi)$. The expression (2.9) is convenient for calculations when $R$ is small, in which case the series converges extremely rapidly.

In the form of definite integrals

$$\frac{u}{U_0} = \operatorname{erf} \frac{1}{2}\eta + J_0 \ \text{for}\ x > 0, \qquad \frac{u}{U_0} = 1 - J_1 \quad \text{for}\ x < 0 \qquad (2.10)$$

where

$$J_n = \frac{1}{(2\pi)^{3/2}} \int_R^{\infty} \left[\left(\frac{z}{z-\eta}\right)^{1/2} - (-1)^n \left(\frac{z}{z+\eta}\right)^{1/2}\right] e^{-\frac{z^2}{8}} K_{1/4}\left(\frac{z^2}{8}\right) dz \quad (n = 0 \text{ или } 1)$$

which are correct everywhere, but are convenient for calculations when $R$ is large.

The results of computations [8] according to formulas (2.9), (2.10) are presented in Fig. 1, where $u/U_0$ is shown as a function of $\xi$ for various values of $\eta$. The curves in Fig. 1 give a graphic picture of the way in which the motion of the fluid along the plate develops in the first instant of time. Thus for $x > 0$ the flow may be divided into two regions. The first region, described approximately by $\xi > 2$, is characterized by the fact that the influence of the leading edge of the plate is small, so that the motion of the fluid corresponds to the flow near a plate infinite in both directions; that is, $u/U_0 = \operatorname{erf} \frac{1}{2}\eta$. In the second region ($0 < \xi < 2$) the action of the leading edge is important in the motion of the fluid, and the variable $x$ enters into the expression for the velocity. From Fig. 1 it is seen that there does not exist any sharp boundary between the regions just described. Nevertheless, we will endeavor tentatively to specify the shape of this boundary by introducing a criterion for the relative influence of the $x$-coordinate (leading edge of the plate) on the longitudinal velocity component $u$:

$$\frac{\partial u}{\partial x} \Bigg| \left(\frac{\partial u}{\partial x}\right)_{x=0}.$$

Then the equation of the boundary may be written down as follows:

$$\frac{\partial u / \partial x}{[\partial u / \partial x]_{x=0}} = \frac{1}{\sqrt{2}} \frac{\xi}{R} \left[\left(\frac{R}{R-\eta}\right)^{1/2} - \left(\frac{R}{R+\eta}\right)^{1/2}\right] \exp\left(-\frac{\xi^2}{8}\right) \frac{K_{1/4}(1/8 R^2)}{K_{1/4}(1/8 \eta^2)} = \varepsilon \qquad (2.11)$$

where $\epsilon$ is a sufficiently small quantity. The dependence of $\eta$ on $\xi$, computed according to formula (2.11) for $\epsilon = 0.05$, is presented in Fig. 2. The nature of this dependence allows the conclusion first of all that the influence of the edge of the plate extends over a larger interval as $\eta$ increases. Furthermore, for large $\eta$ the curve in Fig. 2 approximates to a line $\xi = \text{const.}$, whereas for small $\eta$ it approaches the value $\xi = 0$ with a certain slope. The latter result appears to be a consequence of the fact that for any small $\xi$ not equal to zero, some $\eta \ll \xi$ may be chosen such that the equation (2.11) will be satisfied.

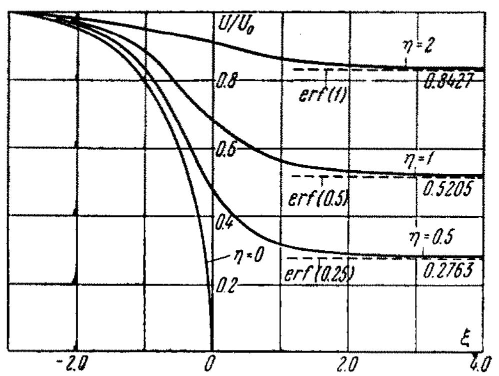


Fig. 1.

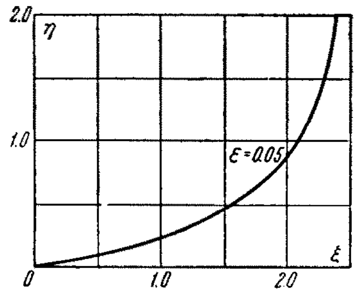


Fig. 2.

From the considerations which have been brought out relative to the two regions of the flow and to the boundary between them, it may be concluded that in the initial period of the motion the influence of the leading edge of the plate will be propagated along the $x$-axis in a region whose width is proportional to $\sqrt{(\nu t)}$. This is evidently explained by the fact that at the very beginning of the motion, when $\sqrt{(\nu t)} \gg U_0 t$, diffusio plays a dominant role in the movement of vorticity along the $x$-axis. In the course of time the quantity $U_0 t$ becomes larger than $\sqrt{(\nu t)}$ and the propagation of vorticity begins to depend on transport by the flow past the plate. During this period the velocity of transport of vorticity reaches the magnitude $U_0$ at the edge of the boundary layer, and consequently the width of the region of influence of the leading edge of the plate for large $y$ is approximately equal to $U_0 t$. For decreasing values of $y$ the width of the region in question decreases because of the retarding action of the plate, and finally for very small $y$ the region of influence in terms of the coordinate $x$ near the plate vanishes altogether. These results correspond to some extent to the approximate solutions presented above for the problem (1.1), (1.2). In fact, the solution (1.4), which is obtained with the aid of methods valid for small $y$, leads to a width equal to $3/8\ U_0 t$ for the region of influence of the leading edge of the plate. On the other hand, the formula (1.5), obtained through the construction of an "outer" solution valid for large $y$, gives the width of this region as $U_0 t$.

It should be observed that the solution (2.9), (2.10), in contrast to known solutions for the problem of the stationary flow past a semi-infini plate [1,6] is accurate for all values of $x$. This raises the possibility of studying certain interesting aspects of the flow near a plate with the aid of (2.9), (2.10). In particular, the influence of the plate on the external flow of fluid situated to the left of the axis $oy$ may be followe using the solution (2.9), (2.10). Thus, if $\xi$ is negative, it is seen from the second formula (2.10) that for large $R$ and finite $\eta$ the dominant term in the asymptotic expansion for $u/U_0$ does not depend on $\eta$, and the curves in Fig. 1 tend to one and the same value. Moreover, substantial changes in velocity ahead of the plate occur mostly in the region $|\xi| < 2$.

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